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STRONG AND WEAK MINIMAL REPAIR, THE REVIVAL PROCESS, AND MAINTAINED SYSTEM RELIABILITY MODELS

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1 INTRODUCTION

1.1 Rationale

The standard minimal repair, or "bad-as-old," stochastic process (Ascher and Feingold, 1984) results from assumptions that force the process to be a Poisson process. This can sometimes be inconvenient for modeling because the requirement that the increments be independent is very strong. We are led to explore weaker versions of the minimal repair assumptions that lead to a stochastic process that has Poisson one-dimensional marginals but does not require independence of the increments. These assumptions are called the weak minimal repair assumptions to distinguish them from the strong minimal repair assumptions commonly used heretofore, and the resulting point process is called a revival process. The weak minimal repair assumptions approximate better the behavior of systems brought back to service by the replacement of a single failed component or subassembly and provide the analyst with a more flexible modeling framework for the reliability of maintainable systems.

1.2 Scope

In this paper, we generalize the "Bad-As-Old" or "Minimal Repair" (BAO/MR, for short) model by positing weaker dependence of the future evolution of the stochastic point process of failure times on its past than obtains in the point process of the BAO/MR model, while retaining the notion that repairs restore the unit to its operating condition just before the failure. With this weaker condition, $N(t)$, the number of events (failures) in the time interval [0. t], still has a Poisson distribution under regularity conditions equivalent to Thompson's

(1981), but independence of the increments is no longer necessary. In the discrete case we also consider, $N(t)$ has the same (multinomial) distribution as that of a sum of independent binomial random variables. We also consider use of this generalized model for maintained systems in which repair is not taken to be instantaneous.

1.3 Detailed Rationale

We have pursued this work primarily for three reasons. First, the weaker conditions we posit admit of a larger class of point processes that retain most of the familiar properties that are commonly invoked in applications. Modelers and statisticians alike may benefit from the greater flexibility afforded by this greater generality. Similarly, for various phenomena, more restrictive models may have been constructed with properties that are either not needed or may indeed hinder effective application. Many applications of the familiar point process models of applied probability involve their one-dimensional properties only; this, a second motive of our study is to show how the use of models that place unwarranted restrictions on higher-dimensional distributions may be avoided. For example, in reliability models, it is often enough to know that the number of failures occurring in a given time interval has a Poisson distribution. Similarly, in insurance risk analysis, it may sometimes be enough to know that that number of claims arriving during a given discrete time interval has a geometric distribution. The higher-dimensional properties, such as independence of the increments of a Poisson process, may not be called for either because the target situation cannot be reasonably assumed to have these properties or because we may simply wish to impose on our model the least possible number of properties that may be irrelevant or difficult to verify. For example, it is not unreasonable to imagine a data set that contains sequences of observed failure times of (identical) maintained units and that has the following two properties: statistical tests for independence of the increments (of the process presumed to be generating the data) fail, while the distribution of the number of failures in every interval of the form $[0, t]$ appears, to all practical statistical purposes, to be Poisson. One could analyze such data with a Poisson process model, but this may be inappropriate because independence of the increments has not been validated. The revival process, introduced in this paper, could provide an appropriate model for this data analysis. Third, and finally, this paper exemplifies an important principle of simplification: we may be able to generalize a stochastic process to one that retains familiar properties (such as Poisson one-dimensional marginals) but dispenses with strong assumptions that nay not apply or may not be verifiable in practical applications. The same idea is developed in the context of two-state models for maintained system reliability in Marlow and Tortorella (1995).

The revival process is an example of appoint process whose one-dimensional distributions are Poisson but who increments are not necessarily independent. We will show below that, for the target situation that is usually used to exemplify minimal repair, the revival process forms a more natural model than the nonhomogeneous Poisson process (NHPP) model that has heretofore been universally invoked.

1.4 Background

In studies of the reliability of a maintained unit, models that posit instantaneous repair and restoration (when a failure occurs) of the unit to its condition immediately before the failure are often used in cases where a renewal model would be inappropriate. A well-known model of this type is the "Bad As Old" or "Minimal Repair" model (Barlow and Hunter (1960), Ascher (1968), Downton (1971), Thompson (1981), Baxter (1982), Ascher and Feingold (1984)). The "Bad As Old" terminology arises from the notion that, for a unit with an NBU life distribution, renewal repair (replacing the unit with a new one or completely overhauling it to brand-new condition) makes the unit "good as new," whereas restoring it to its condition immediately before the failure makes it "bad as old." In the latter instance, the unit's hazard rate as a function of global time (*i. e.*, total time elapsed since the time origin) does not change after the repair (in contrast to the renewal repair scenario in which it is returned to its value at the origin). The "Minimal Repair" terminology arises from the notion the repair performed is only enough to "bring the unit back to life" without affecting any part of it that did not participate in the failure. For example, if the unit is composed of several simple components (*i. e.*, ones that are replaceable but not themselves repairable), then a minimal repair of the unit can be effected by replacing the component(s) that failed with working one(s) that have the same global age.

1.5 Synopsis

We show by elementary means that the weak minimal repair assumptions are sufficient for the existence of a revival process, a point process whose one-dimensional distributions are Poisson but that does not have independent increments. We explore properties of the revival process and study the alternating process made up of a revival process modeling the operating times of some equipment and a renewal process modeling the downtimes of the equipment. Analogues of the familiar asymptotic formulas for an alternating renewal process are obtained.

2 THE STRONG MINIMAL REPAIR PROCESS

To fix notation, let the times between events in a stochastic point process be denoted by $\{U_1, U_2, \ldots\}$, let $S_n = U_1 + \cdots + U_n$, and let $N(t) = \sup\{n : S_n \le t\}$.

Definition. A stochastic point process is said to have the *strong minimal repair* property if

$$P\{U_{n+1} \le t \mid S_1 = s_1, S_2 = s_2, \ldots, S_n = s\} = P\{U_1 \le t + s \mid U_1 > s\} \tag{2.1}$$

for all $t \ge 0$, $0 \le s_1 \le s_2 \le \cdots \le s$, and $n = 1, 2, \ldots$.

The BAO/MR point process has the strong minimal repair property (Barlow and Hunter (1960), Ascher (1968)). Thompson (1981) shows that a BAO/MR process with $EN(t)$ continuous for $t \ge 0$ is equivalent to a NHPP with mean function $EN(t)$, and, further, that if F, the distribution of U_1, has a density f, then

$$EN(t) = \int_0^t f(x)\left[1 - F(x)\right]^{-1} dx = -\log\left[1 - F(t)\right]. \tag{2.2}$$

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Because the NHPP is the limiting process for the superposition of mutually independent, uniformly sparse, regular point processes (Snyder (1973), Thompson (1981)), the BAO/MR model has also been useful as an approximate model for the reliability of a complex system with many failure modes and negligible downtimes (Blumenthal, Greenwood, and Herbach (1976), Holcomb and North (1985)).

3 THE WEAK MINIMAL REPAIR ASSUMPTIONS AND THE REVIVAL PROCESS

3.1 Definition and Basic Properties

The following definition initiates the generalization of the BAO/MR model that we shall study.

Definition. A stochastic point process is said to have the *weak minimal repair* property if

$$P\left\{U_{n+1} \le t \mid S_n = s\right\} = P\left\{U_1 \le t + s \mid U_1 > s\right\} \tag{3.1}$$

for all s, $t \ge 0$ and $n = 1, 2, \ldots$.

In a process that has the strong minimal repair property (2.1), the future development of the process depends on all the previous event times (or, equivalently, on the duration of all the previous times between events). In a process that has the weak minimal repair property (3.1) , the future development of the process depends only on the time of the most recent event (or, equivalently, on the sum of the previous times between events). We will call a process having the weak minimal repair property a *revival process*. Our objective is to obtain a more complete description of revival processes in terms of their event description and time description.

The mathematical distinction between strong and weak minimal repair is expressed through the dissimilarity of (2.1) and (3.1). What is important for modeling, however, is that from an engineering point of view, the distinction turns on the concept of "what the unit remembers about its past history of failure times." This concept of "the unit's memory" may be illustrated in practical reliability modeling terms by considering the following two examples. First, consider a unit made up of a large number of simple replaceable units in a series reliability configuration. Repair of the system is effected by replacing failed components. In the usual interpretation of "minimal repair," every replacement component has the same (global) age as the component it replaces. Let us now look at the stochastic point process of these replacement times at the time S_n of the n^{th} replacement. From the point of view of the future evolution of the process (as described, for example, by the distribution of U_{n+1}), the unit with failures and replaced components is completely indistinguishable from another (identical) copy of the same unit which started operating at the same time origin and *in which no other failures have ever occurred*. We maintain that this is exactly the situation described by weak minimal repair. In mathematical terms, conditioning on the individual values of U_1, U_2, …, U_n is irrelevant and only the conditioning on S_n has any effect. In engineering terms, the unit would "remember" what happened to it at the epochs S_1, S_2, …, only if something had occurred at those epochs to make the unit's future evolution look different from what it would be if nothing had happened at those epochs.

Theorem 1. $\{U_1, U_2, \ldots\}$ is a revival process if the following two conditions are satisfied:

I) The conditional probabilities

$$K_n(t, x) = P\left\{U_{n+1} \le t \mid S_n = x\right\} \tag{3.2}$$

are the same for all n (and so will be denoted by $K(t, x) = K_n(t, x) = K_1(t, x) = P\{U_2 \le t \mid U_1 = x\}$, and

II) The kernel $K(t, x)$ in (3.2) is given by

$$K(t,x) = P\{U_2 \le t \,|\, U_1 = x\} = P\{U_1 \le t + x \,|\, U_1 > x\} = \frac{F(t+x) - F(x)}{1 - F(x)} \tag{3.3}$$

whenever $F(x) < 1$, where $F(x) = P\{U_1 \le x\}$.

Proof. Refer to the definition of weak minimal repair and equation (3.1). ■

Condition I says that when service is restored following a failure, the time to the next failure may depend on the (global) age of the unit at the time of failure, but not on the position of that failure in the sequence of failures.

For a process satisfying only Condition I of Theorem 1, we may obtain an integral equation for the expected number of events in the process by time t. Note that these events need not form a revival process unless Condition II of Theorem 1 is also satisfied.

Theorem 2. Denote $EN(t)$ by $H(t)$. If condition I of Theorem 1 is satisfied, then

$$H(t) = F(t) + \int_0^t K(t - x, x)\, dH(x)\,. \tag{3.4}$$

Proof. Use the identity $P\{N(t) \ge n\} = P\{S_n \le t\}$ and the recursion

$$P\{S_{n+1} \le t\} = \int_0^t P\{S_{n+1} \le t \big| S_n = x\}\, dP\{S_n \le x\} = \int_0^t K(t-x, x)\, dP\{S_n \le x\}\,. \;\blacksquare \tag{3.5}$$

This theorem shows that under Condition I alone, an integral equation similar to the renewal equation can be derived from (3.5) for the expected value of $N(t)$. Of course, if condition II is also satisfied, then $N(t)$ is a revival process and the integral equation is for the expected number of revivals in $[0, t]$.

Note that in (3.3), the kernel K is the right-hand side of (2.1), and so Condition II expresses in a natural way the assumption that revivals restore a unit to operating condition without changing its age. The key difference between a revival process model and the corresponding Bad-As-Old, or strong minimal repair model, is that the full conditioning on the past is not used as it was in (2.1). As we will show below, these processes share some important properties, but there are revival processes that do not satisfy (2.1), so the revival process offers greater generality and flexibility to the modeler.

Now substitute (3.3) into (3.4). Then we obtain

$$H(t) = F(t) + \int_0^t \frac{F(t) - F(x)}{1 - F(x)}\, dH(x)\,,$$

and may then be rewritten as

$$\int_o^t \frac{1}{1 - F(x)}\, dH(x) = \frac{F(t)}{1 - F(t)}\,, \quad F(t) < 1. \tag{3.6}$$

This equation holds for any revival process, but the form of its solution depends on the regularity properties of F. We consider two cases: (i) F has a continuous density f, and (ii) F is purely discrete with jumps at 1, 2, … . In both cases it will be shown that the intensity function of the revival process is the hazard rate of F.

3.2 The Revival Process in Continuous Time

If F has a continuous derivative f, the solution of (3.6) is readily seen to be $H(t) = -\log[1 - F(t)]$, $F(t) < 1$. It is immediate that the process intensity $H'(t) = f(t)[1 - F(t)]^{-1}$, the hazard rate of F.

Evidently, the form of the distribution of $N(t)$ depends on the regularity properties of F. In particular, it follows from (3.3) and (3.4) that, when F is continuous, $N(t)$ has a Poisson distribution with mean $H(t) = -\log[1 - F(t)]$ whenever $F(t) < 1$. Baxter (1982) shows that the same Poisson distribution results if the "relevation transform" (Krakowski (1973)) is applied successively to F. Note that this does not imply that $\{N(t) : t \geq 0\}$ is a Poisson process, which would have been the case had the stronger condition (2.1) been assumed instead of (3.1), (3.2), and (3.3). An example of a continuous-time, mean-continuous revival process that is not a Poisson process is given in Section 3.5.1, showing that conditions (3.1), (3.2), and (3.3) are strictly weaker than (2.1).

3.3 The Revival Process in Discrete Time

Assume F is a lattice life distribution with jumps at 1, 2, … , and let $h(j)$ denote the probability of a revival at time j. Then (3.6) becomes

$$\sum_{j=1}^{t} \frac{h(j)}{P\{U_1 \geq j+1\}} = \frac{P\{U_1 \leq t\}}{P\{U_1 \geq t+1\}}, \quad P\{U_1 \leq t\} < 1, \quad t = 1, 2, \ldots .$$

The solution is

$$h(j) = \frac{P\{U_1 = j\}}{P\{U_1 \geq j\}}, \quad j = 1, 2, \ldots, t .$$

The right-hand side is the hazard rate of F in the discrete case, and it follows that

$$H(t) = \sum_{j=1}^{t} h(j) = \sum_{j=1}^{t} \frac{F(j) - F(j-1)}{1 - F(j-1)}, \quad F(t) < 1 . \tag{3.7}$$

Theorem 3. Conditions (3.1), (3.2), and (3.3) imply that $N(t)$ has the distribution of a sum of independent binomially distributed random variables: for all t = 1, 2, … such that $P\{U_1 \leq t\} < 1$,

$$P\{N(t) = n\} = P\{X_1 + X_2 + \cdots + X_t = n\}, \quad n = 0, 1, \ldots, t , \tag{3.8}$$

where $X_1, X_2, \ldots, X_t$ are mutually independent and

$$P\left\{X_j = 1\right\} = 1 - P\left\{X_j = 0\right\} = h(j), \quad j = 1, 2, \ldots, t. \tag{3.9}$$

Proof. For $j > n + 1 > 2$, we have

$$P\left\{S_{n+1} = j\right\} = \sum_{i=n}^{j-1} P\left\{S_{n+1} = j \mid S_n = i\right\} P\left\{S_n = i\right\}. \tag{3.10}$$

Conditions (3.1), (3.2), and (3.3) imply that, for $j > i + 1$,

$$P\left\{S_{n+1} = j \mid S_n = i\right\} = P\left\{U_1 = j \mid U_1 \geq i + 1\right\} = \frac{P\left\{U_1 = j\right\}}{P\left\{U_1 \geq i + 1\right\}}.$$

Define $Q_n(j) = P\{S_n = j\}/P\{U_1 = j\}$. Then, whenever $F(j) < 1$, the recursion (3.10) may be written

$$Q_1(j) = 1, \quad Q_{n-1}(j) = \sum_{i=n}^{j-1} \frac{h(i)}{1 - h(i)} Q_n(i). \tag{3.11}$$

Multiplying (3.11) by z^{n-1} and summing on n, we obtain

$$P\left\{S_1 = 1\right\} = h(1), \quad \sum_{n=1}^{j} z^{n-1} P\left\{S_n = j\right\} = h(j) \prod_{i=1}^{j-1} \left[1 - (1 - z) h(i)\right].$$

Now sum on j from 1 to t to obtain

$$(1 - z) \sum_{n=1}^{t} z^{n-1} P\left\{S_n \leq t\right\} = 1 - \prod_{i=1}^{t} \left[1 - (1 - z) h(i)\right] = (1 - z) \sum_{n=1}^{t} z^{n-1} P\left\{\sum_{i=1}^{t} X_i \geq n\right\}, \tag{3.12}$$

where $X_1, \ldots, X_t$ are independent and $P\{X_n = 1\} = 1 - P\{X_n = 0\} = h(n)$, $n = 1, 2, \ldots$. The right-hand equality in (3.12) follows by letting $M = X_1 + \cdots + X_t$ and taking expectations on the left and right sides of the identity

$$1 - z^M = (1 - z) \sum_{n=1}^{M} z^{n-1} = (1 - z) \sum_{n=1}^{t} z^{n-1} I\left\{M \geq n\right\}, \quad M \leq t.$$

The theorem now follows from (3.12) and $P\{S_n \leq t\} = P\{N(t) \geq n\}$. ■

3.4 Summary

The results of Sections 3.2 and 3.3 show that the process intensity is unchanged by repair, and this reflects the basic idea Ascher (1968) wished to capture with the Bad-As-Old model. In fact, weak minimal repair is a weaker sufficient condition for invariance of the process intensity; a key result is that independence of the increments of the process $\{N(t) : t \geq 0\}$ is not required for (3.8) and (3.9). An example of a discrete revival process satisfying (3.8) and (3.9) but whose increments are not mutually independent is given in Section 3.5.2.

3.5 Examples

3.5.1 Continuous Time

We first construct a counting process $\{N(t) : t \geq 0\}$ that is not a Poisson process but whose inter-event times satisfy

$$P\{U_{n+1} > t \mid U_1 + \cdots + U_n = x\} = e^{-t}, \quad x, t \geq 0, \quad n = 1, 2, \ldots, \tag{3.13}$$

$$P\{U_1 > t\} = e^{-t}, \quad t \geq 0. \tag{3.14}$$

If F is a continuous life distribution having $F(t) < 1$ for all $t < \infty$, and $H(t) = -\log [1 - F(t)]$, then $\{N(H(t)) : t > 0\}$ is a revival process but not a Poisson process.

Let U_1 and U_2 be independent, each having a negative exponential distribution with mean one. Define $U_3 = -\log [U_1/(U_1 + U_2)]$. Then U_3 has a negative exponential distribution with mean one and is independent of $U_1 + U_2$. Now choose independent and identically distributed U_4, U_5, … such that the vector (U_1, U_2, U_3) and the sequence $\{U_n : n \geq 4\}$ are independent. If $P\{U_4 > t\} = e^{-t}$, $t \geq 0$, then conditions (3.13) and (3.14) are satisfied. If $S_n = U_1 + \cdots + U_n$ and $N(t) = \sup\{n : S_n \leq t\}$, then $\{N(t) : t \geq 0\}$ is not a Poisson process because U_1, U_2, and U_3 are not mutually independent. But, using (3.13) and (3.14), it follows that $N(t)$ has a Poisson distribution with mean t for every $t > 0$.

3.5.2 Discrete Time

Let F be a lattice life distribution supported on the positive integers. Let the hazard rate of F be denoted by

$$h(j) = \frac{F(j) - F(j-1)}{1 - F(j-1)} > 0, \quad j = 1, 2, \ldots . \tag{3.15}$$

Choose X_1, X_2, and X_3 such that $P\{X_j = 1\} = 1 - P\{X_j = 0\} = h(j)$, $j = 1, 2, \ldots$, with X_1 and X_2 independent, X_3 independent of $X_1 + X_2$, and X_1, X_2, and X_3 not mutually independent. Next, let $\{X_n : n \geq 4\}$ be mutually independent, independent of the vector (X_1, X_2, X_3), and having $P\{X_j = 1\} = 1 - P\{X_j = 0\} = h(j)$, $j \geq 4$. Set $N(t) = X_1 + \cdots + X_t$, $t \geq 1$, and $S_n = \sup\{t : N(t) \leq n\}$. Then for $j \geq i + 1 \geq n + 1 \geq 2$,

$$\begin{aligned} P\{S_{n+1} \geq j+1, S_n = i\} &= P\{X_1 + \cdots + X_i = n, X_{i+1} + \cdots + X_j = 0\} \\ &= P\{X_1 + \cdots + X_i = n\} P\{X_{i+1} = 0\} \cdots P\{X_j = 0\}. \end{aligned} \tag{3.16}$$

Note that when $i = 1$ and $j = 3$, independence of $X_1 + X_2$ and X_3 implies that the events $\{X_1 = 0\}$, $\{X_2 = 0\}$, and $\{X_3 = 0\}$ are mutually independent. It follows from (3.16) that, for $n \geq 1$,

$$P\{S_{n+1} \le j \mid S_n = i\} = 1 - \prod_{k=i+1}^{j} [1 - h(k)] = \frac{P\{i+1 \le S_1 \le j\}}{P\{S_1 \ge i+1\}}.$$

Thus $\{S_n : n \ge 1\}$ defines a revival process. By construction, however, $N(t)$ does not have independent increments, even though its distribution for each t would be the same (namely, multinomial) as if $\{X_1, X_2, \dots\}$ were mutually independent.

4 THE INTERRUPTED REVIVAL PROCESS

To illustrate the use of the revival process in modeling repairable systems having nonzero repair times, we next consider a revival process interrupted (Marlow and Tortorella 1995) by a renewal downtime sequence $\{D_n\}$ with $0 < ED_1 < \infty$. Let $D = ED_1$ and denote the distribution of D_1 by G. It follows from the elementary renewal theorem and the strong law of large numbers for renewal sequences that the convergence condition (4.2) of Theorem 1 of Marlow and Tortorella (1995) is satisfied.

We first show that condition (4.1) of Theorem 1 of Marlow and Tortorella (1995) is satisfied provided that the limit $L = \lim_{t\to\infty} EN_U(t)/t$ exists and $0 < L < \infty$. Then, assuming further that the sequences $\{U_n\}$ and $\{D_n\}$ are independent, we show that the conclusions of Theorem 1, parts (i) and (ii), continue to hold even if $L = 0$ or $L = \infty$. As before, both continuous and discrete revival processes will be considered for the uninterrupted failure process.

4.1 The Interrupted Revival Process in Continuous Time

4.1.1 Finite, Non-Zero Asymptotic Hazard Rate

Assume that F has a continuous density and that $F(t) < 1$, $t \ge 0$. Then $N_U(t)$ has a Poisson distribution with mean $H(t)$ given by $H(t) = -\log[1 - F(t)]$,

$$P\{N_U(t) = n\} = \frac{H(t)^n}{n!} e^{-H(t)}, \quad n = 0,1,2,\dots. \tag{4.1}$$

It follows that, for $s > 0$,

$$E\left(e^{-sN_U(t)}\right) = e^{-\left[1-e^{-s}\right]H(t)}. \tag{4.2}$$

Then Karamata's theorem [18] shows that

$$\lim_{n\to\infty} \frac{ES_n}{n} = \lim_{s\to 0^+} s\int_0^\infty e^{-\left[1-e^{-s}\right]H(t)}\, dt \tag{4.3}$$

provided at least one limit exists and is finite. Assume now that $\lim_{t\to\infty} \frac{H(t)}{t} = L$ with $0 < L < \infty$. The right-hand limit in (4.3) is then $1/L$ and

$$\lim_{n\to\infty}\frac{ES_n}{n}=\frac{1}{L}=\lim_{t\to\infty}\frac{t}{H(t)}=\lim_{t\to\infty}\frac{t}{EN_U(t)}.$$

Conditions (4.1) and (4.2) of Theorem 1 of Marlow and Tortorella (1995) are satisfied, and it follows that if $0 < L < \infty$ and $0 < D < \infty$, then

$$\lim_{t\to\infty}\frac{EN(t)}{t}=\frac{L}{1+LD}. \tag{4.4}$$

Lemma 4. $\lim_{t\to\infty}\frac{N_U(t)}{t}=L$ almost surely.

Proof. Because $N_U(t)$ is nondecreasing a. s., it suffices to prove that $N_U(m^2)/m^2$ converges to L a. s. as $m \to \infty$ along the positive integers. We have $EN_U(t) = H(t)$ and, by assumption, $H(t)/t \to L < \infty$ as $t \to \infty$, the proof will be complete if we can show that $[N_U(m^2) - H(m^2)]/m^2 \to 0$ a. s. as $m \to \infty$. By the Borel-Cantelli lemma, it suffices that the series

$$\sum_{n=1}^{\infty}P\left\{\left|N_U(m^2)-H(m^2)\right|>hm^2\right\} \tag{4.5}$$

converges for every $h > 0$. Noting that $N_U(t)$ has a Poisson distribution whose mean and variance are both $H(t)$, application of Chebyshev's inequality shows that the terms of (4.5) are dominated by $H(m^2)]/h^2m^4$, and convergence of the series is assured. ■

By application of Lemma 4 and Theorem 1 of Marlow and Tortorella (1995), we see that

$$\lim_{t\to\infty}\frac{N(t)}{t}=\frac{L}{1+LD}\quad \text{a. s.} \tag{4.6}$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T R(t)\,dt=\frac{1}{1+LD}\quad \text{a. s.} \tag{4.7}$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T A(t)\,dt=\frac{1}{1+LD}. \tag{4.8}$$

Here the sequences $\{U_n\}$ and $\{D_n\}$ are not required to be independent. If they are independent, it can further be shown that (4.4) and (4.8) hold even if $L = 0$ or $L = \infty$ (see Section 4.3).

Summarizing the continuous case, if the mean downtime is finite and F has an "exponential tail" (*i. e.*, $\lim_{t\to\infty}[1-F(t)]^{1/t}=e^{-L}$, $0<L<\infty$, then the asymptotic system failure rate and the long-term average availability are given by (4.4) and (4.8), respectively. If L were the reciprocal of the mean operating time, these would be familiar results for alternating renewal processes. However, for an interrupted revival process with renewal downtimes, L is the long-term hazard rate (when it exists) of F:

$$L = \lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{1}{1-F(x)}\, dF(x).$$

4.1.2 The Case $L = 0$ or $L = \infty$

In this Section, we show that (4.4) and (4.8) hold even when $L = 0$ or $L = \infty$ by obtaining the Laplace-Stieltjes transform of $EN(t)$ and $A(t)$ and passing to the limit. Note first that because $N_U(t)$ has a Poisson distribution with mean $H(t)$,

$$P\{N_U(t) \geq n\} = \int_0^t P\{N_U(x) = n-1\}\, dH(x).$$

Substitution into equation (3.3) of Marlow and Tortorella (1995) yields

$$EN(t) = \sum_{n=1}^{\infty} \int_0^t dG_n(x) \int_0^{t-x} P\{N_U(y) = n-1\}\, dH(y).$$

Writing N^* and G^* for the Laplace-Stieltjes transforms of EN and G, respectively, it follows that because G_n is a convolution,

$$N^*(s) = \sum_{n=0}^{\infty} \left[G^*(s)\right]^n \int_0^{\infty} P\{N_U(x) = n\} e^{-sx} dH(x), \quad s > 0.$$

Substituting the Poisson distribution (4.1) for $N_U(t)$ and summing, the last becomes

$$N^*(s) = \int_0^{\infty} e^{-\left[1-G^*(s)\right]H(t)} e^{-st}\, dH(t).$$

Integration by parts then gives

$$\left[1-G^*(s)\right] N^*(s) = 1 - \int_0^{\infty} e^{-\left[1-G^*(s)\right]H(t/s)} e^{-t}\, dt. \tag{4.9}$$

The Laplace-Stieltjes transform of A follows in a similar manner from equation (3.5) of Marlow and Tortorella (1995):

$$s\int_0^{\infty} A(t) e^{-st} dt = \int_0^{\infty} e^{-\left[1-G^*(s)\right]H(t/s)} e^{-t}\, dt. \tag{4.10}$$

The last two equations have been written so that the limiting forms as $s \to 0^+$ are evident. If $H(t)/t \to L$ as $t \to \infty$, it follows from (4.9) and (4.10) and Karamata's theorem (Widder, 1941) that

$$\lim_{t\to\infty}\frac{EN(t)}{t}=\begin{cases}0, & L=0\\ D^{-1}, & L=\infty\end{cases}\quad \text{and}$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T A(t)\,dt=\begin{cases}1, & L=0\\ 0, & L=\infty\end{cases}$$

where D is the mean of G: $D=\lim_{s\to 0^+}\frac{1-G^*(s)}{s}$.

Some discussion of these cases $L = 0$ and $L = \infty$ is in order. The latter would occur if the hazard rate of F were unbounded as $t \to \infty$, and may be a reasonable model for some complex systems requiring more frequent repair with increasing age. In this case, the proportion of time that the system can be used decreases with time even though revival of failed components continues operation of the system. The case $L = 0$ is perhaps less readily adapted for modeling, but can occur when the hazard rate of F is decreasing. When this is the case, revivals would provide better service than renewals, because in the renewal case the hazard rate of the replacement unit (equivalent to "new") is always greater than that of the failed unit. A class of continuous distributions for which all these possibilities occur is of course the Weibull family.

4.2 The Interrupted Revival Process in Discrete Time

Assume F is purely discrete with jumps at 1, 2, …, and that $F(j) < 1$ for $j \geq 1$. The results just obtained hold in this case also: if the average hazard rate of F has a limit given by

$$L=\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}h(j),\quad 0\le L\le 1,$$

then (4.4) and (4.6) through (4.8) follow.

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APPENDIX 1. Examples of Revival Processes